The Poincaré Group
Algebraic, Representation-Theoretic, and Geometric Aspects

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PLAN

- Generalities
- Representation Theory
- Invariant Wave Equations
- Derivations of P-Group
- Thomas Rotation
- Observer Sky
- Simultaneity
- Rigid Motion

Minkowski-Space and Lorentz-Group

Definition 1 Minkowski space *is the four-dimensional, real, affine space* $\mathbb{A}^4$ *over* $\mathbb{R}^4$*, endowed with a symmetric, non-degenerate, bilinear form* g *of signature* $(+,-,-,-) = (1{,}3)$*. We write* $\mathbb{M}^4 = (\mathbb{A}^4, \mathsf{g})$*.*

Definition 2 *The* (homogeneous) Lorentz group *is the linear group (subgroup of of* $\mathsf{GL}(4,\mathbb{R})$*) of isometries of* $\mathbb{M}^4$*, also called* $\mathsf{O}(1{,}3)$*.*

As topological space $\mathsf{O}(1{,}3)$ decomposes into four connected components. Here $+/-$ stands for positive/negative determinant, $\uparrow$ / $\downarrow$ for time-orientation preserving/reversing:

$$\mathsf{O}(1{,}3) = \underbrace{\mathsf{O}_+^\uparrow(1{,}3) \; \cup \; \mathsf{O}_+^\downarrow(1{,}3)}_{\mathsf{SO}(1{,}3)} \cup \mathsf{O}_-^\uparrow(1{,}3) \; \cup \; \mathsf{O}_-^\downarrow(1{,}3)$$

Of these four components only $\mathsf{O}_+^\uparrow(1{,}3)$ is a subgroup, called the group of proper, orthochronous Lorentz transformations. Elementwise composition with space/time reflections gives $\mathsf{O}_-^\uparrow(1{,}3)/\mathsf{O}_-^\downarrow(1{,}3)$.

The Poincaré Group

For any group $G \subset GL(n, \mathbb{R})$, there is a corresponding *inhomogeneous group* IG, given by the semi-direct product

$$IG = \{(a, A) \mid a \in \mathbb{R}^n\,,\ A \in G\}$$

where

$$(a, A)(a', A') = (a + A \cdot a'\,,\, AA')$$

It can again be thought of as subgroup of $GL(n+1\,,\, \mathbb{R})$ via the embedding

$$(a, A) \longmapsto \begin{pmatrix} 1 & 0^{\dagger} \\ a & A \end{pmatrix}$$

For $G = O(1,3)$ we get the Poincaré group $P := IG$. For $G = O^{\uparrow}_{+}(1,3)$ we set $IG = P^{\uparrow}_{+}$ etc.

Polar Decomposition 1

Theorem 3 *Every element of* $\mathrm{GL}(n,\mathbb{R})$ *is the unique product (depending on choice of order) of a symmetric positive-definite with an orthogonal matrix.*

Applied to $\mathrm{O}_{+}^{\uparrow}(1,3) \subset \mathrm{GL}(4,\mathbb{R})$ this leads to the decomposition of any proper orthochronous Lorentz transformation L into a boost B and a proper spatial rotation R. Let

$$L = \begin{pmatrix} \gamma & \vec{a}^{\dagger} \\ \vec{b} & \mathbf{M} \end{pmatrix}$$

be a Lorentz transformation. This is equivalent to

$$\vec{a}^2 = \gamma^2 - 1\,, \quad \gamma\vec{b} = \mathbf{M}\cdot\vec{a}\,, \quad \mathbf{M}\cdot\mathbf{M}^{\dagger} = \mathbf{1} + \vec{b}\otimes\vec{b}^{\dagger}$$

and correspondingly for $\vec{a} \leftrightarrow \vec{b}$, $\mathbf{M} \leftrightarrow \mathbf{M}^{\dagger}$.

Note: polar decomposing an element in $\mathrm{G} \subset \mathrm{GL}(n,\mathbb{R})$ does not generally lead to factors in G. But this is true for $\mathrm{O}(p,q)$ and $\mathrm{U}(p,q)$.

Polar Decomposition 2

Now, given $L \in O^{\uparrow}_{+}$. Then

$$L = B \cdot R$$

with

$$B = \begin{pmatrix} \gamma & \vec{b}^{\dagger} \\ \vec{b} & \mathbf{1} + \frac{\vec{b}\otimes\vec{b}^{\dagger}}{1+\gamma} \end{pmatrix} \qquad R = \begin{pmatrix} 1 & \vec{0}^{\dagger} \\ \vec{0} & \mathbf{M} - \frac{\vec{b}\otimes\vec{a}^{\dagger}}{1+\gamma} \end{pmatrix}$$

where

$$\begin{aligned} \mathrm{EV}(B) &= (\gamma + \sqrt{\gamma^2 - 1}\,,\, \gamma - \sqrt{\gamma^2 - 1}\,,\, 1\,,\, 1) > 0 \\ \mathbf{D} &= \mathbf{M} - \frac{\vec{b} \otimes \vec{a}^{\dagger}}{1+\gamma} \in \mathsf{SO}(3) \end{aligned}$$

Polar Decomposition 3

Re-express L in terms of parameters $\vec{\beta} := \vec{v}/c$ and $\mathbf{D}$.

Have

$$\gamma = 1/\sqrt{1-\beta^2} \qquad \vec{b} = \gamma\vec{\beta} \qquad \vec{a} = \gamma\mathbf{D}^\dagger \cdot \vec{\beta}$$

so that

$$L(\vec{\beta}, \mathbf{D}) = \underbrace{\begin{pmatrix} \gamma & \gamma\vec{\beta}^\dagger \\ \gamma\vec{\beta} & \mathbf{1} + \frac{\gamma^2}{1+\gamma}\,\vec{\beta}\otimes\vec{\beta}^\dagger \end{pmatrix}}_{B(\vec{\beta})} \underbrace{\begin{pmatrix} 1 & \vec{0}^\dagger \\ \vec{0} & \mathbf{D} \end{pmatrix}}_{R(\mathbf{D})}$$

The polar decomposition defines a topological map from the group onto the cartesian product of its factor-images. Here this leads to the following topological equivalence:

$$\mathsf{O}^\uparrow_+(1,3) \;\cong\; \underbrace{B_1(\vec{0})}_{\text{boosts}} \times \underbrace{\mathsf{SO}(3)}_{\text{rotations}} \;\cong\; \mathbb{R}^3 \times \mathbb{R}\mathsf{P}^3$$

Composition of Boosts 1

Polar decompose the composition of two boosts:

$$B(\vec{\beta}_2) \circ B(\vec{\beta}_1) \;=:\; B(\vec{\beta}_2 \star \vec{\beta}_1) \circ R(\mathsf{T}[\vec{\beta}_2, \vec{\beta}_1])$$

Here $\star$ denotes the operation of *relativistic velocity addition*:

$$\vec{\beta}_2 \star \vec{\beta}_1 \;:=\; \frac{\vec{\beta}_2 + \vec{\beta}_1^{\|} + \gamma_2^{-1}\vec{\beta}_1^{\perp}}{1 + \vec{\beta}_1 \cdot \vec{\beta}_2}$$

$\mathsf{T}[\vec{\beta}_2, \vec{\beta}_1] \in \mathsf{SO}(3)$ is the so called *Thomas rotation* in the $\vec{\beta}_1 - \vec{\beta}_2$ plane with angle φ, where

$$\cos\varphi = \frac{(1 + \gamma + \gamma_1 + \gamma_2)^2}{(1+\gamma)(1+\gamma_1)(1+\gamma_2)} - 1$$

and

$$\gamma \;:=\; \gamma(\vec{\beta}_2 \star \vec{\beta}_1) = \gamma_1\gamma_2(1 + \vec{\beta}_1 \cdot \vec{\beta}_2)$$

Using $\gamma_1 = \cosh\theta_1$ etc., this is just the ‘law of cosines’ for ‘triangles’ on the 4-velocity hyperbola $g(u,u) = 1$.

Composition of Boosts 2

The general law for composing Lorentz transformations is now as follows:

$$\mathrm{L}(\vec{\beta}_2, \mathbf{D_2})\circ\mathrm{L}(\vec{\beta}_1, \mathbf{D_1}) \;=\; \mathrm{L}(\vec{\beta}_2\star \mathrm{D}_2\cdot\vec{\beta}_1\,,\; \mathbf{T}[\vec{\beta}_2, \mathbf{D}_2\cdot\vec{\beta}_1]\cdot\mathbf{D_2}\cdot\mathbf{D_1})$$

Compare with (homogeneous) Galilei transformations:

$$\mathrm{G}(\vec{v}_2, \mathbf{D_2}) \circ \mathrm{G}(\vec{v}_1, \mathbf{D_1}) \;=\; \mathrm{L}(\vec{v}_2 + \mathrm{D}_2 \cdot \vec{v}_1\,,\; \mathbf{D_2} \cdot \mathbf{D_1})$$

where we have the group isomorphism

$$\mathsf{Gal}^{\uparrow}_{+} \cong \underbrace{\mathbb{R}^3}_{\text{boosts}} \rtimes \underbrace{\mathsf{SO}(3)}_{\text{rotations}}$$

Remark 4 *Boosts form an abelian normal subgroup of* $\mathsf{Gal}^{\uparrow}_{+}$. *Hence* $\mathsf{Gal}^{\uparrow}_{+}$ *is not semi-simple. In contrast,* $\mathsf{O}^{\uparrow}_{+}(1,3)$ *has no non-trivial normal subgroup (see below), so it is even simple. This may be seen as due to the Thomas rotation.*

Algebraic Structure of Velocity Space 1

Velocity space ('$\vec{\beta}$–space') is the open ball of radius 1 in $\mathbb{R}^3$. The $\star$ operation makes it a *groupoid*. There is a unique neutral element $\vec{0}$, the *unit*, and each element $\vec{\beta}$ has a unique (left and right) inverse $-\vec{\beta}$:

$$\vec{0} \star \vec{\beta} = \vec{\beta} \star \vec{0} = \vec{\beta}\,, \qquad \vec{\beta} \star (-\vec{\beta}) = (-\vec{\beta}) \star \vec{\beta} = \vec{0}$$

The $\star$ operation is neither commutative nor associative, e.g.

$$\begin{aligned} \vec{\beta}_2 \star \vec{\beta}_1 &= \mathbf{T}[\beta_2, \beta_1] \cdot (\vec{\beta}_1 \star \vec{\beta}_2) \\ \vec{\beta}_3 \star (\vec{\beta}_2 \star \vec{\beta}_1) &= (\vec{\beta}_3 \star \vec{\beta}_2) \star \mathbf{T}[\vec{\beta}_3, \vec{\beta}_2] \cdot \vec{\beta}_1 \end{aligned}$$

Algebraic Structure of Velocity Space 2

However, equations such as

$$\vec{\beta}_1 \star \vec{\beta}_2 = \vec{\beta}_3$$

can still be solved uniquely for $\vec{\beta}_1$ (given $\vec{\beta}_{2,3}$) or $\vec{\beta}_2$ (given $\vec{\beta}_{1,3}$):

$$\begin{aligned}\vec{\beta}_1 &= \vec{\beta}_3 \star (-\mathbf{T}[\vec{\beta}_3, \vec{\beta}_2] \cdot \vec{\beta}_2) \\ \vec{\beta}_2 &= (-\vec{\beta}_1) \star \vec{\beta}_3\end{aligned}$$

This makes the groupoid a *quasi group*. Together with the existence of a unit we get the structure of a *loop*.

This algebraic structure is closely related to hyperbolic geometry (calculus of geodesic segments).

The Universal Cover Group

$$\mathsf{SL}(2,\mathbb{C}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{C};\; ad - bc = 1 \right\}$$

The polar-decomposition map yields the topological homeomorphism

$$\mathsf{SL}(2,\mathbb{C}) \cong \mathbb{R}^3 \times \mathsf{SU}(2) \cong \mathbb{R}^3 \times S^3$$

Let $\{\sigma^\mu\} := \{\mathbf{1}, \vec{\sigma}\}$ and $\{\tilde{\sigma}^\mu\} := \{\mathbf{1}, -\vec{\sigma}\}$. There is a real-analytic 2-1 group homomorphism $\pi : \mathsf{SL}(2,\mathbb{C}) \to \mathsf{O}^{\uparrow}_{+}(1,3)$, given by

$$[\pi(\boldsymbol{A})]^\mu_\nu = \tfrac{1}{2}\mathsf{Trace}(\tilde{\sigma}^\mu \boldsymbol{A} \sigma_\nu \boldsymbol{A}^\dagger)$$

which is essentially the double-cover map $S^3 \to \mathbb{R}\mathsf{P}^3$. A local (no global !) inverse is

$$A = \pm\, \sigma_\mu \, L^\mu_\nu \, \tilde{\sigma}^\nu / \sqrt{\det(\sigma_\mu L^\mu_\nu \tilde{\sigma}^\nu)}$$

For example

$$\begin{aligned}
A_{\mathrm{rot}}(\alpha, \vec{n}) &= \pm \exp(-\tfrac{i}{2}\alpha\, \vec{n}\cdot\vec{\sigma}) \\
A_{\mathrm{boost}}(\beta, \vec{n}) &= \pm \exp(\tfrac{1}{2} \underbrace{\tanh^{-1}(\beta)}_{\text{rapidity}}\, \vec{n}\cdot\vec{\sigma})
\end{aligned}$$

The corresponding Inhomogeneous group (double cover of proper orthochronous Poincare group) is now given by the semi direct product $\mathbb{R}^4 \rtimes_\pi \mathsf{SL}(2\mathbb{C})$

$$\bar{\mathsf{P}}^{\uparrow}_{+} = \{(a, A) \mid a \in \mathbb{R}^4\ , A \in \mathsf{SL}(2, \mathbb{C})\}$$

where

$$(a, A)(a', A') = (a + \pi(A)\cdot a'\ , \ AA')$$

Action on 'Observer Sky'

Identify the space of past null-rays emanating from some event with two-sphere (cross section through past null-cone), and that with Riemann sphere $\mathbb{C}\cup\{\infty\}$ coordinatized by z:

$$k = (-1, \vec{k}) \quad \text{with} \quad \vec{k}^2 = 1\,, \qquad z := \frac{k^1 + ik^2}{1 - k^3}$$

Then $L \in O^{\uparrow}_{+}$ acts on observer sky via

$$z \mapsto \frac{dz - c}{-bz + a}\,, \quad \text{where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \pi^{-1}(L)$$

This is a Möbius transformation with

$$\begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1}$$

These form the group of conformal transformations of S^2 (the observer sky) which acts 3-fold transitive.

Stabilizers of Light Rays

An interesting subgroup of Möbius transformations is that which fixes (stabilizes) a light ray, say $z = \infty$:

$$\mathrm{Stab}(\infty) = \left\{ \mathbf{A}(\delta, d) = \begin{pmatrix} d^{-1} & 0 \\ d^{-1}\delta & d \end{pmatrix} \;\middle|\; d \in \mathbb{C}^{\times}, \delta \in \mathbb{C} \right\}$$

Hence

$$\mathbf{A}(\delta, d) \cdot \mathbf{A}(\delta', d') = \mathbf{A}(\delta + d^2\delta', \, dd')$$

which identifies it as a semi-direct product $\mathbb{C} \rtimes \mathbb{C}^{\times}$. Boosts along and rotations about the 3-axis are parametrized by the modulus and phase of d respectively. For $|d| = 1$ get subgroup that fixes the light ray pointwise. It is obviously isomorphic to the double cover of $\mathsf{E}_2 = \mathbb{R}^2 \rtimes \mathsf{SO}(2)$, that classifies helicity states of massless fields. For $d = 1$ get subgroup isomorphic to $\mathbb{R}^2$, corresponding to certain null rotations:

$$\begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix} = \exp(-\tfrac{i}{2}\,\delta\,\vec{n}\cdot\vec{\sigma})\,, \quad \text{where } \vec{n} = (i, 1, 0)$$

General null rotations:

$$\exp(-\tfrac{i}{2}\vec{v}\cdot\vec{\sigma})\,, \quad \text{with} \quad \vec{v}\cdot\vec{v} = 0\,, \vec{v} \in \mathbb{C}^3$$

In (Q-)Field Theory, null rotations (i.e. the translational part of $\bar{E}(2)$) are represented trivially in order to get finite-dimensional ‘internal’ state spaces. But they do play a rôle in some version of string theory.

Some Facts on $\mathfrak{sl}(2,\mathbb{C})$

Theorem 5 $\mathfrak{sl}(2,\mathbb{C})$ *is simple*

Let $\mathfrak{sl}(2,\mathbb{C})$ be spanned by

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad e_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \qquad [e_+, e_-] = h \qquad [h, e_\pm] = \pm 2e_\pm$$

If $X = ae_+ + be_- + ch \in I =$ ideal, then

$$[e_+, [e_+, X]] = -2be_+ \qquad [e_-, [e_-, X]] = -2ae_-$$

This shows: if any of $a, b, c \neq 0$ then $I = \mathfrak{sl}(2,\mathbb{C})$.

Theorem 6 $\mathfrak{sl}(2,\mathbb{C})$ *has no non-trivial finite dimensional unitary representations.*

Suppose $T : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{u}(n)$, $T(X) =: \hat{X}$ were such a representation. Then $[\hat{h}, \hat{e}_\pm] = \pm 2\hat{e}_\pm$, so that

$$\text{Trace}(\hat{e}_+^2) = \tfrac{1}{2}\text{Trace}(\hat{e}_+[\hat{h}\hat{e}_+ - \hat{e}_+\hat{h}]) = 0$$

$$\Rightarrow \quad \hat{e}_+ = 0 \Rightarrow e_+ \in \ker(T) \Rightarrow \ker(T) = \mathfrak{sl}(2,\mathbb{C})$$

Finite Dim. Representations of $\mathsf{SL}(2,\mathbb{C})$

Theorem 7 *Representations of a connected and simply-connected Lie group are in bijective correspondence to representations of its Lie algebra. This correspondence respects all reducibility properties and equivalences.*

Theorem 8 (Weyl's 'unitarisation trick')
Representations of a real Lie algebra $\mathcal{L}$ on a complex vector space are in bijective correspondence to complex-linear representations of its complexification $\mathcal{L}\otimes\mathbb{C}$ (where $\otimes$ is over $\mathbb{R}$). This correspondence respects all reducibility properties and equivalences, but it does not respect unitarity.

Have

$$\mathfrak{sl}(2,\mathbb{C})\otimes\mathbb{C}\cong(\mathfrak{su}(2)\otimes\mathbb{C})\oplus(\mathfrak{su}(2)\otimes\mathbb{C})$$

Hence the complex, finite-dimensional, irreducible representations of $\mathsf{SL}(2,\mathbb{C})$ are $\mathsf{D}^{(p,q)}$, where (p,q) are independent and $\in\mathbb{N}/2$. None of them is unitary. All representations of $\mathsf{SL}(2,\mathbb{C})$ are *fully* reducible.

Finite Dim. Representations of $SL(2,\mathbb{C})$

'Left' and 'right' complexified $SU(2)$ are complex conjugate to each other. Hence, if $\vee$ denote the symmetrized tensor product, have

$$D^{(p,q)}(\vec{\beta},\vec{\alpha}) = A^{p\vee}(\vec{\beta},\vec{\alpha}) \otimes \bar{A}^{q\vee}(\vec{\beta},\vec{\alpha})$$

so that

$$\Psi^{X_1\cdots X_p}_{Y'_1\cdots Y'_q} \longmapsto \left(A^{X_1}_{Z_1}\cdots A^{X_p}_{Z_p}\right)\,\Psi^{Z_1\cdots Z_p}_{V'_1\cdots V'_q}\,\left(\bar{A}^{V'_1}_{Y'_1}\cdots \bar{A}^{V'_q}_{Y'_q}\right)$$

The Clebsch-Gordan Series for $SL(2,\mathbb{C})$ is

$$D^{(p,q)}\otimes D^{(p',q')} = \bigoplus_{r=|p-p'|}^{p+p'}\ \bigoplus_{s=|q-q'|}^{q+q'} D^{(r,s)}$$

The Pauli Lemmas

The *Pauli-Index* is a map

$$\pi : D^{(p,q)} \to ((-1)^{2p}, (-1)^{2q}) \in \mathbb{Z}_2 \times \mathbb{Z}_2$$

It satisfies

$$\pi(D^{(p,q)} \otimes D^{(p',q')}) = \pi(D^{(p,q)}) \cdot \pi(D^{(p',q')})$$

which says, that it is a homomorphism of semigroups.

According to its representation, we can associate a Pauli-Index to any spinor. For example, a tensor of odd/even degree has PI $(-,-)/(+,+)$.

Now consider the most general linear equation for integer spin (free fields)

$$\sum \partial_{(-,-)} \Psi_{(+,+)} = \sum \Psi_{(-,-)}$$
$$\sum \partial_{(-,-)} \Psi_{(-,-)} = \sum \Psi_{(+,+)}$$

These are invariant under

$$\Theta : \begin{cases} \Psi_{(+,+)}(x) & \mapsto \quad \Psi_{(+,+)}(-x) \\ \Psi_{(-,-)}(x) & \mapsto \; -\Psi_{(-,-)}(-x) \end{cases}$$

Now consider any current, that is a polynomial in the fields and their derivatives:

$$J_{(-,-)} = \sum \Psi_{(-,-)} + \Psi_{(+,+)}\Psi_{(-,-)} + \Psi_{(+,+)}\partial_{(-,-)}\Psi_{(+,+)} + \Psi_{(-,-)}\partial_{(-,-)}\Psi_{(-,-)} + \quad \cdots$$

Then

$$(\Theta J)(x) = -J(-x)$$

Lemma 9 (Pauli) *Charges of conserved currents cannot be definite in any* $\mathsf{SL}(2,\mathbb{C})$ *invariant, free theory of integer spin fields.*

And (almost) likewise

Lemma 10 (Pauli) *Charges of conserved 2nd rank tensors (e.g. energy momentum), bilinear in fields, cannot be definite in any* $\mathsf{SL}(2,\mathbb{C})$ *invariant, free theory of half-integer spin fields.*

Representations of $\bar{\mathsf{P}}^{\uparrow}_{+}$ on Fields

Let V be a representation space for $D^{(p,q)}$ and $\mathcal{F}$ the space of fields $\Psi : \mathbb{R}^4 \to V$. Then $\bar{\mathsf{P}}^{\uparrow}_{+}$ acts on $\mathcal{F}$ as follows, where $g = (a, A)$:

$$(U(g)\Psi)(x) \; = \; D^{(p,q)}(A) \cdot \Psi\big(\pi(A^{-1}) \cdot (x - a)\big)$$

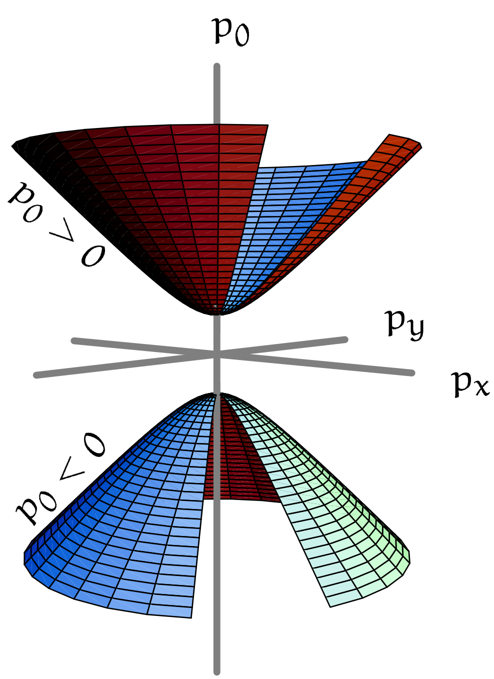
The Fourier transformed version of this is

$$(U(g)\tilde{\Psi})(p) = \exp(ip \cdot a)\, D^{(p,q)}(A) \cdot \tilde{\Psi}\big(\pi(A^{-1}) \cdot p\big)$$

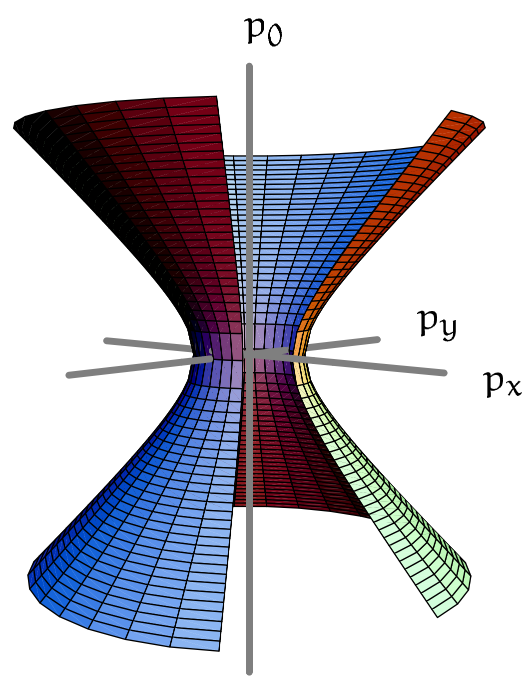
showing that irreducible subspaces must consist of 'functions' $\tilde{\Psi}$ with support contained on $\mathsf{SL}(2, \mathbb{C})$–orbits in momentum space, which are just the hypersurfaces $p \cdot p = m^2$ plus some $p^0 \lessgtr 0$ condition. In spacetime, the first reads:

$$(\Box + m^2)\, \Psi(x) = 0$$

$SL(2,\mathbb{C})$ **Orbits in Momentum Space:** $m^2 \neq 0$

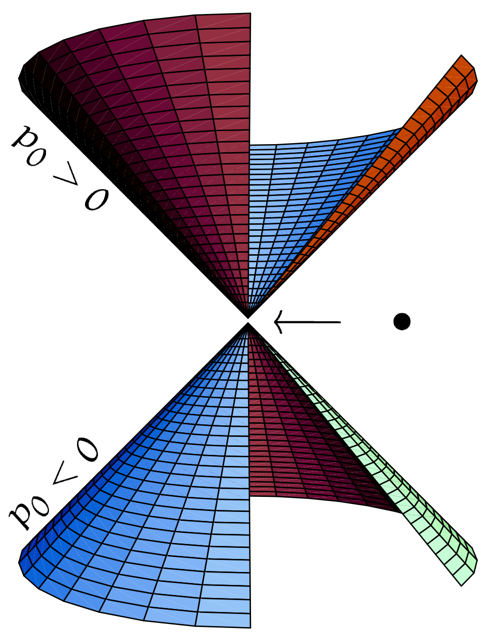


$m^2 > 0$



$m^2 < 0$

$SL(2,\mathbb{C})$ **Orbits in Momentum Space:** $m^2 = 0$



$m^2 = 0$

The one-point orbit $p = 0$ corresponds to a representation where all space-time translations are trivially represented and $D^{(p,q)}$ is reproduced.

Wigner's Trick

Choose a reference point p_* on orbit $\mathcal{O}$ and for each $p \in \mathcal{O}$ an $A_p \in \mathrm{SL}(2,\mathbb{C})$, so that $\pi(A_p) \cdot p_* = p$. Define *Wigner Fields* by

$$\tilde{\Psi}_W(p) := D^{(p,q)}(A_p^{-1}) \cdot \tilde{\Psi}(p)$$

Then

$$(U(A)\tilde{\psi}_W)(p) = D^{(p,q)}\big(W(\pi(A^{-1})\cdot p\,,\,A)\big)\cdot\tilde{\Psi}(\pi(A^{-1})\cdot p)$$

where ('Wigner Rotation')

$$W(p,A) := A^{-1}_{\pi(A)\cdot p} \cdot A \cdot A_p \in \mathcal{S}tab(p_*)$$

This reduces the problem of classifying unitary irreducible representations of $\bar{\mathrm{P}}^{\uparrow}_{+}$ to that of such a classification of $\mathcal{S}tab(p_*)$. Have

$$\mathcal{S}tab(p_*) \cong \begin{cases} \mathrm{SU}(2) & \text{for } p_* \text{ timelike} \\ \bar{\mathrm{E}}(2) & \text{for } p_* \text{ lightlike} \\ \mathrm{SL}(2,\mathbb{R}) & \text{for } p_* \text{ spacelike} \end{cases}$$

A Unifying Viewpoint

All Poincaré invariant linear wave equations in physics, like Klein Gordon, Weyl, Dirac, Maxwell, Rarita-Schwinger, Pauli-Fierz, Bargmann-Schwinger, etc. are projection conditions for the fields to live in irreducible subspaces for $\bar{P}_+^\uparrow$ (or discrete extensions thereof).

Next to $(\Box + m^2)\Psi = 0$ ('momentum irreducibility') this requires the projection condition for 'spin-irreducibility' on the Wigner fields. Let P be the projector onto an $D^{(p,q)}\big|_{Stab(p_*)}$ – irreducible subspace of V. Then the wave equation in momentum space reads

$$\boxed{P\tilde{\Psi}_W = \tilde{\Psi}_W}$$

Translated back to $\tilde{\Psi}$ and then back to Ψ this leads to all the familiar wave equations - and many more!

Notes on Simultaneity: Uniqueness

Definition 11 *A* simultaneity structure *on spacetime* $\mathbb{M}$ *is an equivalence relation on (i.e. a partition of)* $\mathbb{M}$*, such that a physical observer intersects an equivalence class exactly at one point.*

This gives rise to zillions of such structures on any spacetime. Physically, there is no more to require. But it is interesting to know whether there are such structures that respect the automorphism group $\mathrm{Aut}(\mathbb{M})$ of $\mathbb{M}$ or at least the subgroup $\mathrm{Aut}_{\mathsf{X}}$ fixing a structure X (e.g. a single observer, or a field of observers).

Definition 12 *A* fully/restricted invariant simultaneity structure *is a simultaneity structure that is* $\mathrm{Aut}(\mathbb{M})/\mathrm{Aut}_{\mathsf{X}}(\mathbb{M})$ *invariant. In the latter case one speaks of simultaneity relative to* X*.*

Theorem 13 *For* $\mathrm{Aut} = \mathrm{IGal}_{+}^{\uparrow}$ *standard simultaneity is the unique fully invariant simultaneity. For* $\mathrm{Aut} = \mathrm{ILor}_{+}^{\uparrow}$ *no such fully invariant simultaneity exits. If* X = *'inertial observer field', then Einstein simultaneity is the unique restricted simultaneity relative to* X*.*

Notes on Simultaneity
– Uniqueness –

The group theoretic origin of such different behaviour is the following

Theorem 14 *Let* G *be a group with transitive action on a set* S*. Let* $\mathtt{Stab}(p) \subset \mathsf{G}$ *be the stabilizer subgroup for* $p \in \mathsf{S}$ *(due to transitivity, all stabilizer subgroups are conjugate). Then* S *admits a* G*-invariant equivalence relation* $\mathsf{R} \subset \mathsf{S} \times \mathsf{S}$ *(i.e.* $(p, q) \in \mathsf{R} \Leftrightarrow (gp, gq) \in \mathsf{R}$*) iff* $\mathtt{Stab}(p)$ *is* not *maximal, that is, iff* $\mathtt{Stab}(p)$ *is properly contained in a proper subgroup* H *of* G*:* $\mathtt{Stab}(p) \subsetneq \mathsf{H} \subsetneq \mathsf{G}$*.*

For the action of the inhomogeneous Galilei and Lorentz groups on Minkowski space, their stabilizers are the corresponding homogeneous groups. The homogeneous Lorentz group is maximal in the inhomogeneous one. However, the homogeneous Galilei group is properly contained in its semi-direct product with the *spatial* translations, which in turn is properly contained in IGal.

Notes on Simultaneity
– Non-Inertial Observers and Curvature –

Let K be a timelike Killing field on spacetime $\mathbb{M}^4$. Set $g(K, K) = \exp(2\varphi)$ and define

$$A_\mu := K_\mu \exp(-2\varphi)$$

This defines a connection on $\mathbb{M}$, considered as principal $\mathbb{R}$ ('time') bundle over 'space' $S = \mathbb{M}/\text{orbits}(K)$.

Remark 15 *Einstein synchronisation along a curve γ in space S is equivalent to parallel transportation with respect to the connection A.*

Holonomy for the connection A is directly related to the Sagnac effect. The curvature $F = dA$ is proportional to the vorticity of K.

The spatial metric h of S follows from

$$g = \exp(2\varphi)\, A \otimes A - h$$

Notes on Simultaneity
– Non-Inertial Observers and Curvature –

The Riemannian curvature of (S, h) and the bundle curvature $F = dA$ are intimately related, e.g. through the following

Theorem 16 *Let* $R^{(3)}$ *and* Δ *be the Ricci scalar and Laplacian of* (S, h)*. Then the spatial curvature of* S *and the 'time curvature'* F *are related by the Kaluza-Klein identity:*

$$R^{(3)} = 2\big(\Delta\varphi + \|\nabla\varphi\|_h^2\big) - \tfrac{1}{4}\exp(2\varphi)\ \|F\|_h^2$$

Derivations of the Poincaré Group

Usual inputs in derivations are

- Principle of relativity
- Constancy of the speed of light
- Homogeneity and isotropy of space
- Various regularity assumptions:
  - continuity
  - differentiability
  - bijectivity
  - sometimes linearity straightaway
- Reciprocity properties
- Causality properties
- Often hidden assumptions, e.g. concerning space reflections

Derivation of the Poincaré Group

Recall the strongest statement concerning euclidean isometries:

Theorem 17 (Beckman & Quarles 1953)
Consider euclidean space $\mathbb{R}^n$, *where* $n \geq 2$. *Let* $r > 0$ *be some real number and* $f : \mathbb{R}^n \to \mathbb{R}^n$ *a map satisfying*

$$\|x - y| = r \;\Rightarrow\; \|f(x) - f(y)\| = r$$

Then f *is a euclidean motion, i.e.* $f \in \mathsf{E}(3)$.

Note that no assumption concerning continuity or bijectivity of f was made.

Is there an analogous theorem for Minkowski spaces ?

Derivation of the Poincaré Group
– Causality Properties –

Theorem 18 (Alexandrov 1950, Zeeman 1964)
Let $f : \mathbb{M}^n \to \mathbb{M}^n$, $n \geq 3$, *be a bijection so that*

$$\begin{aligned} \textit{either} \quad \|x - y\|_g &= 0 \Leftrightarrow \|f(x) - f(y)\|_g = 0 \\ \textit{or} \quad \|x - y\|_g &> 0 \Leftrightarrow \|f(x) - f(y)\|_g > 0 \end{aligned}$$

then f *is the composition of a Poincaré transformation with a constant dilatation* $x \mapsto \lambda x$, $\lambda \in \mathbb{R}_+$.

There are various generalizations by Alexandrov.

Derivation of the Poincaré Group – Relativity Principle –

We ask: what is the most general group (why a group ?) of maps of space-time transformations, compatible with the principle of relativity (PoR) ?

1 In particular, we take the PoR to mean that the maps should be *automorphisms* of the space-time structure.

2 The space-time structure consists of

2.1 a fixed set of events; hence the maps must be bijections

2.2 an affine structure; e.g. derived from the law of inertia

2.3 homogeneity of space and time and isotropy of space

2.1 and 2.2 imply that transformations must form a subgroup of $\mathsf{Aff}(4,\mathbb{R}) = \mathbb{R}^4 \rtimes \mathsf{GL}(4,\mathbb{R})$ (no continuity assumption !) . The homogeneity part of 2.3 implies that the subgroup contain *all* translations, hence it must be of the form $\mathbb{R}^4 \rtimes \mathsf{G}$. So we seek $\mathsf{G} \subset \mathsf{GL}(4,\mathbb{R})$.

Derivation of the Poincaré Group
– Relativity Principle –

The isotropy part of 2.3 implies that G contains a copy $SO(3)$, which ensures isotropy in one reference system. Isotropy in other system is then expressed by the conjugate subgroups.

3 Concerning boosts we make the following further assumptions:

3.1 Boost transformations can be faithfully parametrized by $\vec{v} \in B_c(\vec{0})$, where $c \in \mathbb{R}_+ \cup \{\infty\}$.

3.2 The map $\varphi : B_c(\vec{0}) \to B_c(\vec{0})$, defined by

$$B(\varphi(\vec{v})) = [B(\vec{v})]^{-1}$$

is continuous.

3.3 Isotropy of space is taken to mean that

$$R(\mathbf{D}) \cdot B(\vec{v}) \cdot R(\mathbf{D}^{-1}) = B(\mathbf{D} \cdot \vec{v})$$

Derivation of the Poincaré Group
– Relativity Principle –

Theorem 19
(Ignatowsky 1910, Frank & Rothe 1911, Berzi & Gorini 1968, ...)
Given the assumptions stated, the homogeneous group G *is either the Galilei group, or the Lorentz group for some finite upper-bound velocity* c.

No continuity assuption was made in deriving linearity. But such an assumption was made for φ in order to derive $\varphi(\vec{v}) = -\vec{v}$ (Berzi-Gorini's contribution). Note however that the inversion map is continuous for topological groups, so that continuity of φ is guaranteed if $\vec{v}$ is a continuous chart.

Rigid Motion

The notion of perfect rigidity (of a physical body) is incompatible with a finite upper bound for all physical signal velocities. But this does not mean that we cannot rigidly move a body, by acting on each of its points with a suitable (point varying) force.

Definition 20 (Born 1909) *A body moves* locally rigid*, if its local rest-frame geometry is unchanging in the course of its motion.*

Mathematically, a motion is described by a timelike vector field X. According to Born, X generates a rigid motion, iff

$$L_X(P_\perp g) = P_\perp(L_X g) = 0$$

where $P_\perp$ is the local projection perpendicular to X. Hence, for $n := X/\|X\|_g$

$$(\delta^a_c - n^a n_c)(\delta^b_d - n^b n_d)(\nabla_a X_b + \nabla_b X_a) \;=\; 0$$

This is clearly implied by, but strictly weaker than, the condition for X to be Killing.

Rigid Motion
– Rotational Motions –

Remark 21 X *generates a rigid motion iff it is free of shear and expansion.*

An interesting task is to classify all *proper* rigid motions, i.e. those which are not isometries.

Theorem 22 (Noether & Herglotz, Part 1) *There are no proper rotational rigid motions in Minkowski space.*

The proof is surprisingly difficult.

Hence you cannot set a body into rotation without deforming it. This is in contrast to translatory motions. You can boost a body rigidly, if you push/accelerate the trailing end harder than you draw/accelerate the leading end (view the orbits of the boost Killing field). If you apply the same forces/accelerations to both ends, the body eventually breaks.

It remains the task to classify all proper non-rotational rigid motions. There are many!

Rigid Motion
– Irrotational Motions –

Take a single timelike curve γ and draw through each of its points the perpendicular hypersurface. Now draw all timelike lines which intersect these hyperplanes perpendicularly. (This works in a tubular neighbourhood of γ where no two hyperplanes intersect.) These flow lines clearly defines a rigid motion, the so called *hyperplane motions*.

Theorem 23 (Noether & Herglotz, Part 2) *Any irrotational proper rigid motion in Minkowski space is a hyperplane motion.*

The proof is fairly easy.